FINITE PLANES WITH LESS THAN EIGHT POINTS ON A LINE.

By C. R. MAC INNES, Princeton University.

We get a finite projective plane if we have a finite number of points arranged in sets, called lines, and subject to the following conditions:

1. If A and B are two distinct points there is one and only one line joining them.
2. If α and β are two distinct lines there is one and only one point common to both.
3. There are at least three points on each line.

It follows from these, that we must have the same number of points on all lines. For, having two lines α and β, we can take a point not on either and join it to all the points of α. Each of these joins must also cut β and we therefore have the points on α and β paired.

Also, there must be the same number of points on a line as there are lines through a point. For, having any point A, take a line β not passing through A; join A to each of the points on β. Each point of β has a line through A and each line through A has a point on β.

In short, then, if we have $n+1$ points on a line, we have $n+1$ lines through a point; n^2+n+1 points in the plane, and n^2+n+1 lines in the plane.

Thus, if we have only three points on a line, we have seven points altogether. Denoting these by 0, 1, 2, ..., 6, we get the lines by the following cyclic scheme:

$$\begin{array}{ccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6; \\ 1, & 2, & 3, & 4, & 5, & 6, & 0; \\ 3, & 4, & 5, & 6, & 0, & 1, & 2; \end{array}$$

the points in the columns being on a line.

If we leave out one line, we have a plane that might be called a finite Euclidean plane. In this, any two points determine a line, and through any point one and only one line may be drawn not meeting a given line. We have n points on each line, $n+1$ lines through each point, and the lines "parallel" in sets of n. An example of this for $n=3$ is the following: Denote a point by a_{ij} and write the points in the form

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{array}$$

Then the rows, columns, and terms in the ordinary determinant expansion give the twelve lines of the plane.

The existence* of such planes has been proved for the case $n=p^m$, p being prime and m any integer. It has also been proved that there is only one distinct type in each of the cases $n=2$, 3, and 4. Our present problem is for $n=5$ and $n=6$.

Having twenty-five points we are to get all possible Euclidean planes built from them. We may choose any set of parallels and write the points in rows, each row being the five points on one of the lines chosen. By properly choosing the order of the points in the rows, we may fix the columns to give another set of parallels. Finally, by properly choosing the order of the rows we may put into the main diagonal the points of some other line. We now have the scheme:

1.1	1.2	1.3	1.4	1.5
2.1	2.2	2.3	2.4	2.5
3.1	3.2	3.3	3.4	3.5
4.1	4.2	4.3	4.4	4.5
5.1	5.2	5.3	5.4	5.5

in which eleven lines are given by the rows, columns, and main diagonal. We know three of the lines through each point of the main diagonal; to find the others. The only possibilities for lines through 1.1 are

1.1	3.2	2.3	5.4	4.5	
		4.3	5.4	2.5	
		5.3	2.4	4.5	
. .	. .	. .	. .	. .	
	4.2	2.3	5.4	3.5	
		5.3	2.4	3.5	(A).
			3.4	2.5	
. .	. .	. .	. .	. .	
	5.2	2.3	3.4	4.5	
		4.3	2.4	3.5	
			3.4	2.5	

Of these we must pick three; they must be one of the following sets:

1.1	3.2	2.3	5.4	4.5	
	4.2	5.3	2.4	3.5	
	5.2	4.3	3.4	2.5	
. .	. .	. .	. .	. .	
	3.2	2.3	5.4	4.5	
	4.2	5.3	3.4	2.5	
	5.2	4.3	2.4	3.5	
. .	. .	. .	. .	. .	(B).

*Veblen and Bussey, *Transactions Mathematical Society*, April, 1906.

3.2	4.3	5.4	2.5
4.2	5.3	2.4	3.5
5.2	2.3	3.4	4.5
. .	. .	. .	. .
3.2	5.3	2.4	4.5
4.2	2.3	5.4	3.5
5.2	4.3	3.4	2.5

Doing the corresponding thing for the other points of the main diagonal, we have four other tables matching (B). We have now to pick a set of three from each table so that the fundamental assumptions are satisfied. This can be done in six ways. This gives twenty-six lines; the remaining four are parallel to the diagonal and can be written down immediately as they are determined uniquely by the others. Apparently there are six planes in this case; they can, however, be brought into a one-to-one correspondence, and again we have only one type. The method of establishing the correspondence analytically is not without interest.

We have denoted points by two numbers. If we think of these as coordinates of the points, we find that one of the six planes is such that the points of a line satisfy a first degree equation, using modulus 5 in the algebra. Thus $x=k$ gives the rows, $x-y=0$ the diagonal, and $x-y=k$ the lines parallel to the diagonal. And so on. The lines of this plane are, of course, interchanged by linear substitutions, of which there are 12,000. If to the plane we apply the transformations

$$x=3x_1^3+3\ldots(1),$$

$$x=3x_2^3+3x_2^2+x_2-1\ldots(2),$$

$$x=4x_3^3+2x_3^2+2x_3+3\ldots(3),$$

$$x=x_4^3-x_4^2+2x_4-1\ldots(4),$$

$$x=x_5^3+2x_5^2+3x_5\ldots(5).$$

y being subject to transformations of the same form, we get the five other planes. These transformations, though not linear, are birational, (1) and (2) being inverse to each other, and each of the remaining three being its own inverse. These five, with the identity, form a group.

If we consider the projective plane, it will be unaltered by linear fractional transformations. These form the simple group L.F(3.5) of order 372,000. This order is 31 times as large as in the Euclidean plane, as there are 31 lines, any one of which might be neglected to give the Euclidean plane from the projective one.

If we now try a similar thing for a plane with 36 points, the work proceeds as before till we have written down all the possibilities for lines through (1, 1), (2, 2), and (3, 3). The others need not be considered.

From the lines through (1, 1) we choose a set of four; to these add four through (2, 2) agreeing with them. This can be done in a number of ways. To the eight so found, it is impossible to add four through (3, 3) so that the fundamental assumptions hold. The first assumption breaks down, no matter what combination be tried.

There is therefore no Euclidean plane with 36 points; and consequently no projective plane with only seven points on a line. This includes the result for $n=7$ given in the answer to problem 142, p. 108, Vol. XIV. It is also the result given by Dr. F. H. Safford in answer to problem 132, p. 215, Vol. XIII.

ON CONSTRUCTING A CUBE HAVING A GIVEN RATIO TO A GIVEN CUBE.*

By R. D. CARMICHAEL.

The semi-cubical parabola is capable of a beautiful application to the problem of finding a cube having a given ratio to a given cube. The object of this paper is to give a method of constructing this curve by continuous motion, to apply the locus to the above problem, and also to show how to construct a line numerically equal in length to the cube root of a given line.

The equation of this curve,

(1) $$x^3=py^2,$$

when transformed to polar coordinates with the axis of x as the polar axis, may readily be reduced to

(2) $$\rho\cos\theta=p\tan^2\theta.$$

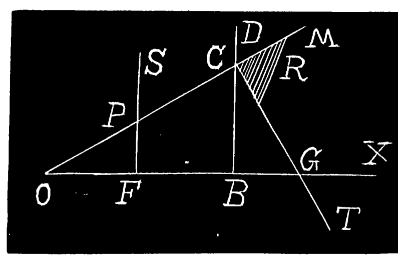

Fig. 1.

To find a construction of the curve by continuous motion we proceed as follows: Let OX (Fig. 1) be the polar axis. Take a distance $OB=p$ and erect BD perpendicular to OX, cutting OM at C, the angle MOX being equal to θ. Draw CG perpendicular to OM at C, intersecting OX in G. From O lay off $OF=BG$, and draw FS perpendicular to OX, intersecting OM in P. P is a point on the semi-cubical parabola.

Proof. Since $OB=p$, $BC=p\tan\theta$. But, since OCG is a right triangle and CB is perpendicular to OG, $OB:BC::BC:BG$. Hence,

*Read before the American Mathematical Society, April, 1907.